Journal of Statistical Physics, Vol. 60, Nos. 3/4, 1990

Master Equations for Subordinated Processes

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Received October 5, 1989; final February 8, 1990

We study Markov jump processes constructed by subordination of diffusion processes. The procedure can be viewed as a randomization or a coarse graining of time. We construct the master equation for the cases of finite and infinite total jump rates, and give a collection of explicitly solvable examples.

KEY WORDS: Markov jump processes; master equation; Levy flight.

1. INTRODUCTION

Markovian stochastic processes are used to model a wide range of physical, biological, and engineering phenomena.(1–3) Diffusion processes governed by Fokker–Planck equations are quite well understood, and there are many explicitly solved examples.(4) Much less is known for processes with jumps, in particular when there is an infinite number of jumps in a finite time interval. In the present work we discuss a method that allows us to construct Markovian processes with jumps, for which one can determine explicitly the transition probability and many other properties. The processes are constructed by subordination of diffusion processes. This procedure, introduced originally by Bochner,(5,7) can be interpreted as a randomization or a coarse graining of time. We will characterize the constructed processes by master equations. For each known diffusion process we can construct families of jump processes with known properties. In particular, models subordinated to the Wiener and the repulsive Wong

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0022-4715/90/0800-0413$06.00/0

processes can be characterized in great detail. The construction also provides models with combined diffusion and jumps. This method was used by Hongler[8,9] to construct exactly solvable jump processes.

Assuming some regularity conditions, the transition probability density $p(x, t \mid x_0)$ of a homogeneous Markov process (with values in the reals) satisfies the following integrodifferential equation (Kolmogorov–Feller equation):

$$\frac{\partial}{\partial t} p(x, t \mid x_0) = -\frac{\partial}{\partial x} [D_1(x)\, p(x, t \mid x_0)] + \frac{\partial^2}{\partial x^2} [D_2(x)\, p(x, t \mid x_0)]$$

$$+ ⨍ dy [W(x \mid y)\, p(y, t \mid x_0) - W(y \mid x)\, p(x, t \mid x_0)] \tag{1.1}$$

where $D_1(x)$ is the drift, $D_2(x)$ is the diffusion coefficient, $W(x \mid y)$ is the jump rate density, and $⨍$ denotes the Cauchy principal part of the integral. Through subordination we will obtain either processes containing the three components—drift, diffusion, and jumps—or pure jump processes, with finite or infinite total rate. The construction also provides a simple way to express the master equation in the form of a Kramers–Moyal expansion.

The paper is structured as follows. In Section 2 we describe the procedure of subordination and give some exaples of subordinators. In Section 3 we discuss general properties of processes subordinated to diffusion processes, referring in particular to their master equations. In Sections 4 and 5 we treat the particular cases of subordinating the Wiener and the repulsive Wong processes. In the Appendices we give the proofs and discuss some more technical aspects.

2. SUBORDINATED PROCESSES

We will summarize the method of subordination of Markov processes, which was developed by S. Bochner. A complete discussion can be found in refs. 7 and 10. The main idea is the following. We start with a time-homogeneous Markov process with transition probability density (t.p.d.) $p(x, t \mid x_0)$, and define a new function $q(x, t \mid x_0)$ by

$$q(x, t \mid x_0) = \int_0^\infty d\gamma_t(s)\, p(x, s \mid x_0) \tag{2.1}$$

where $d\gamma_t(s)$ is a measure that depends parametrically on t and is called the subordinator measure. If $d\gamma_t(s)$ satisfies certain conditions, $q(x, t \mid x_0)$ will be the t.p.d. of a new Markov process. The required conditions can be stated in the following three equivalent forms:

A. $\gamma_t(s)$ is the transition probability distribution of a process with positive independent increments (i.e., a positive, infinitely divisible process)[11]: for all t, $\gamma_t(s)$ is a nondecreasing function of s such that (i)

$$\int_0^\infty d\gamma_t(s) = 1 \qquad \text{for all} \quad t \geqslant 0 \tag{2.2}$$

$$\gamma_t(0) = 0 \qquad \text{for all} \quad t \geqslant 0 \tag{2.3}$$

$$\gamma_{t=0}(0^+) - \gamma_{t=0}(0) = 1 \qquad \text{i.e.,} \quad d\gamma_{t=0}(s) = \delta_+(s)\, ds \tag{2.4}$$

where $\delta_+(s)$ denotes a Dirac delta function defined on the positive reals:

$$\delta_+(s)\, ds \equiv d\theta_+(s), \qquad \theta_+(s) \doteq \begin{cases} 0 & s = 0 \\ 1 & s > 0 \end{cases} \tag{2.5}$$

and (ii) it satisfies the Chapman–Kolmogorov equation on the positive reals:

$$\gamma_{t+r}(s) = \int_{v=0}^{s} \gamma_t(s-v)\, d\gamma_r(v) \tag{2.6}$$

We use the notation $f(0^+) \doteq \lim_{s\to 0^+} f(s)$, where the limit is taken through positive values of s.

Property (i) allows one to interpret the subordination as a randomization of time: One looks at the process at random times s with a probability distribution given by $\gamma_t(s)$. When we discuss the relation with a master equation we will see that it can also be viewed as a coarse graining of time.

Property (ii) states the Markovicity of the subordinator and guarantees that the new process is also Markovian.

B. The distribution $\gamma_t(s)$ can be characterized by its Laplace transform, which, since $d\gamma_t(s)$ is positive, is a completely monotone function for all t. The conditions (i) and (ii) are expressed by requiring that it can be represented as

$$\int_0^\infty d\gamma_t(s)\, e^{-sz} = e^{-tv(z)} \qquad \text{for} \quad z \geqslant 0, \quad t \geqslant 0 \tag{2.7}$$

where $v(z)$ is a continuous function satisfying

$$\text{for } z = 0: \qquad v(0) = 0 \tag{2.8}$$

$$\text{for } z > 0: \qquad v(z) > 0 \text{ and is smooth } (\mathscr{C}^\infty) \tag{2.9}$$

and its derivative is a completely monotone function:

$$(-1)^{n+1} \frac{d^n}{dz^n} v(z) \geqslant 0, \qquad n = 1, 2,... \tag{2.10}$$

We will refer as subordinator interchangeably to the process, the measure $d\gamma_t(s)$, and the function $v(z)$.

C. It can be shown[7] that the most general subordinator $v(z)$ can be represented as

$$v(z) = cz + \lim_{\delta \to 0^+} \int_\delta^\infty d\varrho(s)(1 - e^{-sz}) \tag{2.11}$$

where $c \geqslant 0$ is a constant. The condition (2.10) is equivalent to requiring that $d\varrho(s)$ is positive on $(0, \infty)$. Furthermore, $d\varrho(s)$ must satisfy the following integrability condition at 0 and at ∞:

$$\int_{0^+}^1 d\varrho(s)\, s + \int_1^\infty d\varrho(s) < \infty \tag{2.12}$$

As shown in Appendix B, $d\varrho(s)$ is the jump rate of the subordinator, since it is related to $d\gamma_t(s)$ by the following weak limit:

$$d\varrho(s) = \lim_{t \to 0^+} \frac{1}{t}\, d\gamma_t(s) \qquad \text{in} \quad (0^+, \infty) \tag{2.13}$$

Further, the constant c is its drift, and it does not have a diffusion part. We distinguish two types of jump processes: those with a finite and those with an infinite total jump rate, which, according to (2.7) and (2.11) (for $c = 0$), can be expressed as

$$\int_{0^+}^\infty d\varrho(s) = v(\infty) \qquad \text{and} \quad e^{-tv(\infty)} = \gamma_t(0^+) - \gamma_t(0) \tag{2.14}$$

The subordinator is called proper if the $q(x, t \mid x_0)$ obtained by subordination to a process with a density $p(x, t \mid x_0)$ is also a density. This is the case when $q(x, t \mid x_0)$ is measurable and the following equivalent conditions are satisfied:

$$\text{(a)} \qquad \lim_{z \to \infty} v(z) = \infty \tag{2.15}$$

$$\text{(b)} \qquad \gamma_t(0) = \gamma_t(0^+) \tag{2.16}$$

$$\text{(c)} \qquad \int_{0^+}^1 d\varrho(s) = \infty, \qquad \text{or } c \neq 0 \tag{2.17}$$

A nonproper subordinator thus has a finite total rate and therefore its distribution can be written as[10]

$$d\gamma_t(s) = e^{-tv(\infty)}\delta_+(s)\, ds + [1 - e^{-tv(\infty)}]\, d\gamma_t^{(+)}(s) \tag{2.18}$$

In this case, the subordinated $q(x, t\,|\,x_0)$ consists of a density plus a term with a delta function (see Section 3).

A proper subordinator with $c=0$ has infinite total rate. If $c\neq 0$, the subordinator is proper, independent of the type of jump. As we will see, the type of jump is transmitted to the subordinated process.

A useful property is that if $\nu_1(z)$ and $\nu_2(z)$ are both subordinators, then

$$\nu_2(\nu_1(z)) \qquad [\text{and, as a consequence, } \nu_2(\beta z), \text{ for } \beta>0] \tag{2.19}$$

and

$$\alpha_1\nu_1(z)+\alpha_2\nu_2(z) \qquad \text{for} \quad \alpha_1, \alpha_2>0 \tag{2.20}$$

are also subordinators.[7] Furthermore, we will see that the existence of moments of the new process depends on the differentiability properties of $\nu(z)$ at $z=0$. It is useful to introduce the concept of *regularized subordinator*: To any subordinator $\nu(z)$ we can associate a family of subordinators $\nu_\beta(z)$ depending on a parameter $\beta\geqslant 0$ defined by

$$\nu_\beta(z) \doteq \nu(z+\beta)-\nu(\beta) \tag{2.21}$$

$\nu_\beta(z)$ is again a subordinator, since it satisfies the conditions (2.8)–(2.10). For $\beta>0$ it is smooth ($\mathscr{C}^\infty$) at $z=0$. It is easy to verify using (2.7) that the subordinator measure associated to the regularized $\nu_\beta(z)$ is given by

$$d\gamma_{\beta,t}(s)=d\gamma_t(s)\,e^{t\nu(\beta)}e^{-s\beta} \tag{2.22}$$

Examples

We will discuss a family of subordinators that can be represented in the general form

$$\nu(z)=\frac{\kappa}{\alpha}\left[(\beta+\vartheta z)^\alpha-\beta^\alpha\right] \tag{2.23}$$

where α, β, $\vartheta>0$ and $\kappa>0$ are constants with ranges

$$\begin{aligned} &(\text{I}) \qquad \beta=0, \qquad\quad 0<\alpha\leqslant 1\\ &(\text{II}) \qquad \beta>0, \qquad -\infty<\alpha\leqslant 1 \end{aligned} \tag{2.24}$$

κ and ϑ are time scale normalizations that we will use only to get simpler expressions in the examples by choosing them appropriately.

Case I consists of all stable processes with positive increments. The case $\alpha=0$, $\beta>0$ is to be interpreted as the limit $\alpha\to 0$ of (2.23), giving

$$v(z)=\kappa\ln\left(1+\frac{\vartheta z}{\beta}\right) \tag{2.25}$$

Case II for $\alpha>0$ is a regularization of I. Negative values of α lead to nonproper subordinators.

The corresponding representation (2.11) is: (1) if $\alpha=1$ and $\beta\geqslant 0$

$$c=\kappa\vartheta, \qquad \varrho=0 \tag{2.26}$$

(2) If $0<\alpha<1$ and $\beta\geqslant 0$, or $-\infty<\alpha<1$ and $\beta>0$,

$$c=0, \qquad d\varrho(s)=\frac{\kappa\vartheta^{\alpha}}{\Gamma(1-\alpha)}\, s^{-\alpha-1}e^{-s\beta\vartheta}\, ds \tag{2.27}$$

where $\Gamma(x)$ is the gamma function.

The subordinator measure $d\gamma_t(s)$ is known only for some special values of α:

(i) $\beta\geqslant 0$:

$$\alpha=1: \qquad d\gamma_t(s)=\delta_+(s-\kappa\vartheta t)\, ds \tag{2.28}$$

(ii) $\beta=0$:

$$\alpha=\frac{1}{2}: \qquad d\gamma_t(s)=\kappa t\left(\frac{\vartheta}{\pi s^3}\right)^{1/2} e^{-\kappa^2\vartheta t^2/s}\, ds \tag{2.29}$$

$$\alpha=\frac{1}{3}: \qquad d\gamma_t(s)=\frac{\vartheta^{1/2}}{3\pi}\left(\frac{3\kappa t}{s}\right)^{3/2} K_{1/3}\left(2\left(\frac{\kappa^3\vartheta t^3}{s}\right)^{1/2}\right) ds \tag{2.30}$$

where $K_\nu(r)$ is a modified Bessel function.[12]

(iii) $\beta>0$: The measures for $\alpha=1/2$ and $\alpha=1/3$ can be constructed immediately from (2.29), (2.30), and (2.22). Furthermore,

$$\alpha=0: \qquad d\gamma_t(s)=\left(\frac{\beta}{\vartheta}\right)^{kt}\frac{s^{kt-1}}{\Gamma(\kappa t)}\, e^{-\beta s/\vartheta}\, ds \tag{2.31}$$

$$a=-1: \qquad d\gamma_t(s)=e^{-t\kappa/\beta}\delta_+(s)\, ds+\left(\frac{t}{\vartheta s}\right)^{1/2} e^{-t\kappa/\beta-s\beta/\vartheta} I_1\left(2\left(\frac{ts}{\vartheta}\right)^{1/2}\right) ds \tag{2.32}$$

where $I_1(r)$ is a hyperbolic Bessel function.[12] This last example is a nonproper subordinator.

3. SUBORDINATION OF DIFFUSION PROCESSES

Since the subordinated process is Markovian, its t.p.d. $q(x, t \mid x_0)$ must satisfy an integrodifferential equation of the type (1.1).[3] In Appendix A we prove the following result: If the t.p.d. $p(x, t \mid x_0)$ of the original diffusion process satisfies the Fokker–Planck equation

$$-\frac{\partial}{\partial t} p(x, t \mid x_0) = \frac{\partial}{\partial x} [D_1(x)\, p(x, t \mid x_0)] - \frac{1}{2} \frac{\partial^2}{\partial x^2} [D_2(x)\, p(x, t \mid x_0)] \doteq L_F p \tag{3.1}$$

then the t.p.d. of the process subordinated with a proper subordinator satisfies

$$-\frac{\partial}{\partial t} q(x, t \mid x_0) = c \left\{ \frac{\partial}{\partial x} [D_1(x)\, q(x, t \mid x_0)] - \frac{1}{2} \frac{\partial^2}{\partial x^2} [D_2(x)\, q(x, t \mid x_0)] \right\} + ⨍_{-\infty}^{\infty} dy [W(y \mid x)\, q(x, t \mid x_0) - W(x \mid y)\, q(y, t \mid x_0)] \tag{3.2}$$

The symbol $⨍$ denotes the principal part of the integral. The jump rate $W(x \mid x_0)$ is given by

$$W(x \mid x_0) = \int_{0_+}^{\infty} d\varrho(s)\, p(x, s \mid x_0) \qquad (\text{for} \quad |x - x_0| > \varepsilon > 0) \tag{3.3}$$

and the constant c and the measure $d\varrho(s)$ are those defined in the representation (2.11) of $v(z)$.

Remarks. 1. Equation (3.2) implies that the nature of the trajectories of the subordinated process is determined exclusively by the subordinator (i.e., it is independent of the initial diffusion process): If one chooses $c = 0$ in (2.11), one gets a pure jump process. If $c \neq 0$, $d\varrho(s) \neq 0$, the subordinated process has jumps and diffusion. If $c \neq 0$, $d\varrho(s) = 0$, one recovers the original diffusion with a rescaled time.

2. The expression (3.3) for the jump rate allows us to give an intuitive interpretation to subordination (for $c = 0$): In a time unit, the transitions through jumps correspond to a weighted average of the underlying diffusion. Thus, by looking only at a coarse-grained time, the accumulated diffusion appears as jumps.

3. By subordination we can construct processes that involve both jumps and diffusion. The t.p.d. $q^{(c)}(x, t \mid x_0)$, solution of (3.2) for $c \neq 0$,

can be obtained from the t.p.d. $q^{(0)}(x, t \mid x_0)$ of the pure jump process [i.e., setting $c=0$ in (3.2)] and the one of the original diffusion process as

$$q^{(c)}(x, t \mid x_0) = \int_{-\infty}^{\infty} dy\, p(x, ct \mid y)\, q^{(0)}(y, t \mid x_0)$$

$$= \int_{-\infty}^{\infty} dy\, q^{(0)}(x, t \mid y)\, p(y, ct \mid x_0) \tag{3.4}$$

This result is deduced in Appendix B.

4. Equations (3.1) and (3.2) can be written formally as

$$-\frac{\partial}{\partial t} p(x, t \mid x_0) = \varphi_0(x)\, H \left[\frac{p(x, t \mid x_0)}{\varphi_0(x)}\right] \tag{3.1a}$$

and

$$-\frac{\partial}{\partial t} q(x, t \mid x_0) = \varphi_0(x)\, v(H) \left[\frac{q(x, t \mid x_0)}{\varphi_0(x)}\right] \tag{3.2a}$$

where H is the self-adjoint (Schrödinger) operator defined by

$$H = \frac{1}{\varphi_0(x)} L_F \varphi_0(x) = -\frac{1}{2} \frac{\partial}{\partial x} D_2(x) \frac{\partial}{\partial x} + V(x) \tag{3.5}$$

and $\varphi_0^2(x) \doteq p_{\mathrm{st}}(x)$ is a stationary solution of (3.1). This representation is easily obtained using formal eigenfunction expansions of H. We discuss two examples in Section 4.

3.1. Nonproper Subordinators

The total jump rate of the process constructed with a nonproper subordinator can be expressed using (3.3) and (2.11) as

$$\lim_{\varepsilon \to 0} \int_{|x - x_0| > \varepsilon} dx\, W(x \mid x_0) = \int_{-\infty}^{\infty} dx \int_{0^+}^{\infty} d\varrho(s)\, p(x, s \mid x_0)$$

$$= \int_{0^+}^{\infty} d\varrho(s) = v(\infty) \tag{3.6}$$

i.e., if the subordinator has a finite total rate $v(\infty)$, it will also be the case for the subordinated process, independent of the original diffusion.

A proper subordinator will lead to a process with jumps of the same type as its own.

It is well known[10] that if the total rate is finite, the process typically stays at a fixed position between jumps during a finite time interval. In our case, the t.p.d. constructed with a nonproper subordinator $\nu(z)$ can be written in the form

$$\begin{aligned} q(x, t \mid x_0) &= \int_0^\infty d\gamma_t(s)\, p(x, s \mid x_0) \\ &= [\gamma_t(0^+) - \gamma_t(0)]\, p(x, 0 \mid x_0) + \int_{0^+}^\infty d\gamma_t(s)\, p(x, s \mid x_0) \\ &= e^{-t\nu(\infty)}\delta(x - x_0) + [1 - e^{-t\nu(\infty)}]\, q^{(+)}(t, x, x_0) \end{aligned} \tag{3.7}$$

i.e., $q(x, t \mid x_0)$ consists of a density $q^{(+)}(t, x, x_0)$ plus a delta function. Before each jump the system stays at a fixed position x_0 for a time interval governed by an exponential distribution with mean decay time $1/\nu(\infty)$.

The fact that if $c \neq 0$ the subordinator is always proper has the intuitive interpretation that, if the process has a diffusion or a drift part, it cannot stay at a fixed position for a finite time interval.

Since $q(x, t \mid x_0)$ is not a density, the conditions we require for the construction of (3.2) (see Appendix A) are not satisfied. One can, however, easily construct the master equation for the probability distribution $Q(A, t \mid x_0)$ for transitions from a point x_0 into a set A: Equation (3.7) becomes

$$Q(A, t \mid x_0) = e^{-t\nu(\infty)}\chi_A(x_0) + \int_{0^+}^\infty d\gamma_t(s) \int_A dx\, p(x, s \mid x_0) \tag{3.8}$$

where $\chi_A(x_0)$ is the indicator function of the set A. For a small time Δt we can write

$$\begin{aligned} Q(A, \Delta t \mid x_0) = e^{-\Delta t\nu(\infty)}\chi_A(x_0) + \Delta t \int_{0^+}^\infty d\varrho(s) \\ \times \int_A dx\, p(x, s \mid x_0) + r(A, \Delta t, x_0) \end{aligned} \tag{3.9}$$

with a remainder of $o(\Delta t)$:

$$\begin{aligned} r(A, \Delta t, x_0) = \Delta t \int_{0^+}^\infty \left[d\gamma_{\Delta t}(s) \frac{1}{\Delta t} - d\varrho(s) \right] \\ \times \int_A dx\, p(x, s \mid x_0) = o(\Delta t) \end{aligned} \tag{3.10}$$

due to (2.13). This is the condition required to apply Feller's result[13] to obtain

$$\frac{\partial}{\partial t} Q(A, t \mid x_0) = \int_{-\infty}^{\infty} \left[\int_A dx\, W(x \mid y) \right] Q(dy, t \mid x_0)$$

$$- \nu(\infty) \int_A Q(dx, t \mid x_0) \tag{3.11}$$

with

$$W(x \mid x_0) = \int_{0^+}^{\infty} d\varrho(s)\, p(x, s \mid x_0) \tag{3.12}$$

3.2. Stationary States

Assuming that the diffusion process (3.1) has a stationary state $p_{\mathrm{st}}(x)$ that satisfies detailed balance, we can show that the subordinated process has the same function $p_{\mathrm{st}}(x)$ as stationary state [i.e., $q_{\mathrm{st}}(x) = p_{\mathrm{st}}(x)$] and it also satisfies detailed balance. The condition of detailed balance can be expressed as

$$p(x, t \mid y)\, p_{\mathrm{st}}(y) = p(y, t \mid x)\, p_{\mathrm{st}}(x) \tag{3.13}$$

The definition (2.1) implies that

$$q(x, t \mid y)\, p_{\mathrm{st}}(y) = q(y, t \mid x)\, p_{\mathrm{st}}(x) \tag{3.14}$$

so we have to verify, using this relation, that $p_{\mathrm{st}}(x)$ is a time-independent solution of (3.2), i.e.,

$$⨍_{-\infty}^{\infty} dy [W(y \mid x)\, p_{\mathrm{st}}(x) - W(x \mid y)\, p_{\mathrm{st}}(y)] = 0$$

which follows immediately from (3.14) and $W(x \mid y) = \lim_{t \to 0^+} (1/t)\, q(x, t \mid y)$.

4. SUBORDINATION OF WIENER PROCESSES

One can characterize quite generally the processes subordinated to the Wiener process by an arbitrary subordinator $\nu(z)$. It follows from the definition (2.1) that since the Wiener process is homogeneous in space [i.e., $p(x, t \mid x_0) = p(x - x_0, t \mid 0)$], the subordinated process will also be so. We recall that the t.p.d. $p(x, t \mid x_0)$ and the characteristic function Φ of the Wiener process are given, respectively, by

$$p(x, t \mid x_0) = (2\pi t)^{-1/2} \exp \frac{-(x - x_0)^2}{2t} \tag{4.1}$$

and

$$\Phi(u, t) \doteq \int_{-\infty}^{\infty} dx[\exp(-iux)]\, p(x, t \mid x_0) = \exp\left(\frac{t}{2} u^2 - ix_0 u\right) \tag{4.2}$$

The characteristic function of the subordinated process Φ_v is obtained immediately from Eqs. (2.1), (2.7), and (4.2) as

$$\Phi_v(u, t) = \exp[-tv(u^2/2) - ix_0 u] \tag{4.3}$$

which leads to

$$q(x, t \mid x_0) = \frac{1}{\pi} \int_0^{\infty} du \cos[(x - x_0)\, u]\, e^{-tv(u^2/2)} \tag{4.4}$$

The equation for the time evolution can thus be formally written as

$$-\frac{\partial}{\partial t} q(x, t \mid x_0) = v\left(-\frac{1}{2}\frac{\partial^2}{\partial x^2}\right) q(x, t \mid x_0) \tag{4.5}$$

which is a special case of (3.2a). We remark that the subordinated process can be decomposed into a sum of two independent processes, one a Wiener process and the other a pure jump process. This is easily seen by looking at the characteristic function (4.3), which can be factorized using (2.11) as

$$\Phi_v(u, t) = \exp(-tcu^2/2 - ix_0 u) \exp\left\{-\int_{0^+}^{\infty} \varrho(dr)[1 - \exp(-ru^2/2)]\right\} \tag{4.6}$$

The representation (3.2)–(3.3) gives us the interpretation of the second factor as the characteristic function of a pure jump process.

4.1. Moments

If the characteristic function is $n + 1$ times continuously differentiable in a neighborhood of zero, then the first n moments exist.[14] Thus, for processes subordinated to a Wiener process the existence of the moments depends on the differentiability of the function $v(z)$ in the neighborhood of zero. If $v(z)$ is analytic, all the moments exist. If we apply this criterion to the family of subordinators (2.23), we obtain the following results:

1. For $\beta > 0$ all the moments exist.

2. For $\beta = 0$: If $1/2 < \alpha < 1$, the first moment is zero and all other moments do not exist; if $\alpha \leqslant 1/2$, no moments exist.

In the case where $v(z)$ is analytic, we immediately obtain a development of the form

$$-\frac{\partial}{\partial t} q(x, t \mid x_0) = \sum_{k=1}^{\infty} \frac{(-1)^k}{2^k k!} \eta_k \frac{\partial^{2k}}{\partial x^{2k}} q(x, t \mid x_0) \tag{4.7}$$

with

$$\eta_k = \frac{\partial^k}{\partial z^k} v(z)\bigg|_{z=0} = \begin{cases} c + \int_{0^+}^{\infty} \varrho(dr)\, r & \text{for} \quad k = 1 \\ (-1)^{k-1} \int_{0^+}^{\infty} \varrho(dr)\, r^k & \text{for} \quad k \geqslant 2 \end{cases} \tag{4.8}$$

which is indeed the Kramers–Moyal expansion of the subordinated process (see Appendix C).

4.2. Examples

We will denote the subordinated process by $\{\zeta(t)\}_{t \geqslant 0}$ and the conditional moments by

$$\int_{-\infty}^{\infty} dx (x - x_0)^n\, q(x, t \mid x_0) \doteq \langle [\zeta(t) - x_0]^n \rangle_{x_0}$$

I. As a first example, we consider the Wiener process subordinated with $v(z) = \kappa \vartheta^\alpha z^\alpha / \alpha$ with $\vartheta = 2$ and $\kappa = \alpha$.

It is well known[7] that this choice of $v(z)$ leads to the symmetric *stable Lévy processes* of index 2α.[15] The t.p.d. is given by

$$q(x, t \mid x_0) = \frac{1}{\pi} \int_0^{\infty} du\, e^{-tu^{2\alpha}} \cos(u \mid x - x_0 \mid) \tag{4.9}$$

These processes have applications in the study of the renormalization group,[16] relaxation in polymers,[17,18] and anomalous diffusion.[19] For some special choices of α the integral (4.9) can be expressed in terms of special functions:

(i) $\alpha = 1/2$ leads to the *Cauchy process:*

$$q(x, t \mid x_0) = \frac{t}{\pi[t^2 + (x - x_0)^2]}$$

(ii) $\alpha = 1/3$ gives

$$q(x, t \mid x_0) = \frac{\exp[-2t^3/27(x-x_0)^2]}{2[27\pi(x-x_0)^2]^{1/2}} W_{-1/2, 1/6}\left(\frac{4t^3}{27(x-x_0)^2}\right)$$
$$= \frac{2^{1/3} t^3}{(3\pi)^{1/2} |x - x_0|^{7/3}} G\left(\frac{7}{6}; \frac{4}{3}; \frac{4t^3}{27(x-x_0)^2}\right)$$

where $W_{\mu,\nu}(z)$ is a Whittaker function and $G(a; b; z)$ is a confluent hypergeometric function, both functions being bounded when z tends to infinity.

(iii) $\alpha = 1/4$ leads to

$$q(x, t \mid x_0) = \left(\frac{t^2}{2\pi |x-x_0|^3}\right)^{1/2} \left\{\cos\left(\frac{t^2}{4|x-x_0|}\right)\left[\frac{1}{2} - C\left(\frac{t^2}{4|x-x_0|}\right)\right]\right.$$
$$\left. + \sin\left(\frac{t^2}{4|x-x_0|}\right)\left[\frac{1}{2} - S\left(\frac{t^2}{4|x-x_0|}\right)\right]\right\}$$

where $S(z)$ and $C(z)$ are the sine and cosine Fresnel integrals.

We can easily compute the jump rate for all $\alpha \in (0; 1)$ using the formula (3.3):

$$W(x \mid x_0) = \frac{\Gamma(2\alpha + 1)\sin(\pi\alpha)}{\pi |x - x_0|^{2\alpha + 1}}$$

As we have seen from (3.2) and $c = 0$, these processes consist exclusively of jumps for all α. It is interesting to contrast this fact with the dependence on α of the *fractal Hausdorff–Besicovitch dimension* H_f of the trajectories in the $(t, \zeta(t))$ plane[20]:

$$H_f = \begin{cases} 1 & \text{if } \ 0 < \alpha \leqslant \dfrac{1}{2} \\ 2 - \dfrac{1}{2\alpha} & \text{if } \ \dfrac{1}{2} < \alpha \leqslant 1 \end{cases}$$

with probability one. We finally remark that since the Lévy processes do not have moments,[4] they do not admit a *Kramers–Moyal expansion.* This follows from the fact that $\nu(z) = z^\alpha$ is not analytic at zero.

[4] Except the first moment, if $1/2 < \alpha < 1$.

II. Next we give an example of a process with combined jumps and diffusion, by taking $\nu(z) = cz + z^{1/2}$. This leads to a characteristic function

$$\Phi_\nu(u, t) = \exp(-t(u^2/2) - t|u| - iux_0)$$

and a t.p.d.

$$q(x, t \mid x_0) = \frac{1}{2(2\pi t)^{1/2}} \left[\exp\left[\frac{(t - i|x - x_0|^2}{2t}\right] \operatorname{erfc}\left(\frac{t - i|x - x_0|}{(2t)^{1/2}}\right) + \exp\left[-\frac{(t + i|x - x_0|)^2}{2t}\right] \operatorname{erfc}\left(\frac{t + i|x - x_0|}{(2t)^{1/2}}\right)\right]$$

where $\operatorname{erfc}(z) = 1 - \operatorname{erf}(z)$, and $\operatorname{erf}(z)$ is the normalized error function.[21] This process is the independent sum of a Wiener and a Cauchy process. Its jump rate is that of the Cauchy process:

$$W(x \mid x_0) = \frac{1}{\pi(x - x_0)^2}$$

This process has no moments and hence no Kramers–Moyal expansion.

III. As a third example, we take the subordinator $\nu(z) = \kappa/\alpha[(\beta + \vartheta z)^\alpha - \beta^\alpha]$, choosing the constants $\beta > 0$, $\vartheta = 2/\beta$, and $\kappa = |\alpha| \beta^\alpha$ for $\alpha \neq 0$, and $\kappa = 1$ for $\alpha = 0$. The t.p.d. is then given for $\alpha \in [0; 1)$ by the integral

$$q(x, t \mid x_0) = \frac{\exp(t\beta^{2\alpha})}{\pi} \int_0^\infty du \exp[-t(\beta^2 + u^2)^\alpha] \cos(u|x - x_0|) \tag{4.10}$$

and the jump rate is

$$W(x \mid x_0) = \frac{\alpha}{\sqrt{\pi}\,\Gamma(1-\alpha)} \left(\frac{2\beta}{|x - x_0|}\right)^{\alpha + 1/2} K_{\alpha + 1/2}(\beta|x - x_0|)$$

Since $\nu(z)$ is an analytic function in the neighborhood of $z = 0$, we can write a *Kramers–Moyal development*. Using the result of Appendix C, we obtain

$$\eta_n = \begin{cases} 0 & \text{for } n \text{ odd} \\ \dfrac{\alpha\Gamma(n/2 - \alpha)\, n!}{\Gamma(n/2 + 1)\,\Gamma(1 - \alpha)\,\beta^{n - 2\alpha}} & \text{for } n \text{ even} \end{cases}$$

(i) For the value $\alpha = 1/2$ the integral (4.10) can be evaluated explicitly:

$$q(x, t \mid x_0) = \frac{e^{\beta t}\beta t}{\pi[t^2 + (x - x_0)^2]^{1/2}} K_1(\beta(t^2 + (x - x_0)^2)^{1/2})$$

The conditional moments can also be computed for $\alpha = 1/2$:

$$\langle [\zeta(t) - x_0]^n \rangle_{x_0} = \begin{cases} 0 & \text{for } n \text{ odd} \\ \dfrac{2e^{\beta t}}{\pi} \left(\dfrac{2}{\beta}\right)^{(n-1)/2} \Gamma((n+1)/2)\, K_{(n-1)/2}(\beta t) & \text{for } n \text{ even} \end{cases}$$

(ii) For $\alpha = 0$, the expressions become simpler: the integral (4.10) gives

$$q(x, t \mid x_0) = \frac{\beta^{2t}}{\pi} \int_0^\infty du(\beta^2 + u^2)^{-t} \cos(u\, |x - x_0|)$$

$$= \frac{\beta}{\sqrt{\pi}\,\Gamma(t)} \left(\frac{\beta}{2}\, |x - x_0|\right)^{t-1/2} K_{t-1/2}(\beta\, |x - x_0|)$$

The jump rate becomes

$$W(x \mid x_0) = \frac{e^{-\beta |x - x_0|}}{|x - x_0|}$$

The conditional moments and the Kramers–Moyal coefficients η_n are

$$\langle [\zeta(t) - x_0]^n \rangle_{x_0} = \begin{cases} 0 & \text{for } n \text{ odd} \\ \dfrac{\Gamma(n/2 + t)}{\beta^n \Gamma(t)} \dfrac{n!}{\Gamma(n/2 + 1)} & \text{for } n \text{ even} \end{cases}$$

$$\eta_n = \begin{cases} 0 & \text{for } n \text{ odd} \\ \dfrac{2(n-1)!}{\beta^n} & \text{for } n \text{ even} \end{cases}$$

5. SUBORDINATION OF REPULSIVE WONG PROCESSES

The *repulsive Wong process* is a diffusion process with drift $D_1(x) = \mu\, \text{th}(a; \mu x)$ and diffusion coefficient $D_2 = 1$. Its t.p.d. is given by

$$p(x, t \mid x_0) = \frac{\text{ch}(a; \mu x)}{\text{ch}(a; \mu x_0)}\, e^{-\mu^2 t/2}\, \frac{e^{-(x - x_0)^2/2t}}{(2\pi t)^{1/2}}$$

Its Fokker–Planck equation is

$$-\frac{\partial}{\partial t} p(x, t \mid x_0) = \frac{\partial}{\partial x} [\mu\, \text{th}(a; \mu x)\, p(x, t \mid x_0)] - \frac{1}{2} \frac{\partial^2}{\partial x^2} p(x, t \mid x_0)$$

where we have introduced, for $a \geqslant 0$, the notations $\mathrm{ch}(a; z) = (e^z + ae^{-z})/2$ and $\mathrm{th}(a; z) = \mathrm{ch}(-a; z)/\mathrm{ch}(a; z)$.

This process belongs to a class of exactly solvable models that can be constructed by modifying the spectrum of the Wiener process.[22,23] The Wong process has the following (conditional) characteristic function:

$$\Phi(u, t) = e^{-tu^2/2 - iux_0} \frac{\mathrm{ch}[a; \mu(x_0 - iut)]}{\mathrm{ch}(a; \mu x_0)}$$

If the subordinator is analytic in a neighborhood of zero, we obtain the following (conditional) characteristic function:

$$\Phi_\nu(u, t) = \frac{e^{-iux_0}}{2\,\mathrm{ch}(a; \mu x_0)} \left(e^{\mu x_0 - t\nu(u^2/2 + i\mu u)} + ae^{-\mu x_0 - t\nu(y^2/2 - i\mu u)}\right)$$

and hence all the moments exist. The t.p.d. of the subordinated process is given by the integral

$$q(x, t \mid x_0) = \frac{\mathrm{ch}(a; \mu x)}{\mathrm{ch}(a; \mu x_0)} \frac{1}{\pi} \int_0^\infty du\, e^{-t\nu(u^2 + \mu^2)/2} \cos(u\,|x - x_0|) \qquad (5.1)$$

which can be formally seen as the solution of the time-evolution equation

$$-\frac{\partial}{\partial t} q(x, t \mid x_0) = \mathrm{ch}(a; \mu x)\, \nu \left(-\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{\mu^2}{2}\right) \frac{q(x, t \mid x_0)}{\mathrm{ch}(a; \mu x)}$$

The jump rate is given by

$$W(x \mid x_0) = \frac{\mathrm{ch}(a; \mu x)}{\mathrm{ch}(a; \mu x_0)} \frac{1}{(2\pi)^{1/2}} \int_{0+}^\infty d\varrho(s)\, e^{-\mu^2 s/2 - (x - x_0)^2/2s}$$

The Kramers–Moyal coefficients can be expressed as an integral [at least when $\int_{0+}^\infty d\varrho(s)\, s^{1/2} < \infty$]: For n odd

$$\eta_n(x_0) = \frac{n!\,\mathrm{th}(a; \mu x_0)}{(2\pi)^{1/2}} \int_{0+}^\infty d\varrho(s)\, s^{n/2} e^{-\mu^2 s/4} [(\mathscr{D}_{-n-1}(-\mu\sqrt{s}) - \mathscr{D}_{-n-1}(\mu\sqrt{s})] + c\mu\,\mathrm{th}(a; \mu x_0)\, \delta_{1,n}$$

and for n even

$$\eta_n(x_0) = \frac{n!}{(2\pi)^{1/2}} \int_{0+}^\infty d\varrho(s)\, s^{n/2} e^{-\mu^2 s/4} [\mathscr{D}_{-n-1}(-\mu\sqrt{s}) + \mathscr{D}_{-n-1}(\mu\sqrt{s})] + c\delta_{2,n}$$

where $\mathscr{D}_\nu(z)$ is a parabolic-cylinder function, and $\delta_{j,l}$ is the usual Kronecker symbol.

Examples

We take again the subordinator $v(z)=\kappa/\alpha[(\beta+\vartheta z)^\alpha-\beta^\alpha]$ with $\beta>0$, $\vartheta=2/\beta$, and $\kappa=|\alpha|\ \beta^\alpha$ for $\alpha\neq 0$ and $\kappa=1$ for $\alpha=0$. The case where $\beta=0$ is recovered as a trivial limit, so we shall make only some remarks concerning its particularities when necessary. The t.p.d. is given for $\alpha\in[0;1)$ by the integral

$$q(x,t\,|\,x_0)=\frac{\mathrm{ch}(a;\mu x)}{\mathrm{ch}(a;\mu x_0)}\frac{e^{t\beta^{2\alpha}}}{\pi}\int_0^\infty du\, e^{-t(\beta^2+\mu^2+u^2)^\alpha}\cos(u\,|x-x_0|) \tag{5.2}$$

and the jump rate by

$$W(x\,|\,x_0)=\frac{\alpha 2^{\alpha+1/2}}{\sqrt{\pi}\,\Gamma(1-\alpha)}\frac{\mathrm{ch}(a;\mu x)}{\mathrm{ch}(a;\mu x_0)}\left(\frac{(\beta^2+\mu^2)^{1/2}}{|x-x_0|}\right)^{\alpha+1/2}\times K_{\alpha+1/2}((\beta^2+\mu^2)^{1/2}\,|x-x_0|)$$

The conditional moments and the Kramers–Moyal expansion exist if $\beta>0$, due to the analyticity of $v(z)$ at zero, but not for $\beta=0$. The Kramers–Moyal coefficients for $\beta>0$ also have a general expression in terms of special functions:

$$\eta_n(x_0)=\begin{cases}\dfrac{\alpha 2^{n+1}}{\sqrt{\pi}\,\Gamma(1-\alpha)}\Gamma\left(\dfrac{n}{2}+1\right)\Gamma\left(\dfrac{n+1}{2}-\alpha\right)\mu\,\mathrm{th}(a;\mu x_0) & \\ \quad\times(\beta^2+\mu^2)^{\alpha-(n+1)/2}\,{}_2F_1\left(\dfrac{n}{2}+1,\dfrac{n+1}{2}-\alpha;\dfrac{3}{2};\dfrac{\mu^2}{\beta^2+\mu^2}\right) & \text{for } n \text{ odd}\\ \dfrac{\alpha 2^{n}}{\sqrt{\pi}\,\Gamma(1-\alpha)}\Gamma\left(\dfrac{n+1}{2}\right)\Gamma\left(\dfrac{n}{2}-\alpha\right) & \\ \quad\times(\beta^2+\mu^2)^{\alpha-n/2}\,{}_2F_1\left(\dfrac{n+1}{2},\dfrac{n}{2}-\alpha;\dfrac{1}{2};\dfrac{\mu^2}{\beta^2+\mu^2}\right) & \text{for } n \text{ even}\end{cases}$$

where ${}_2F_1(a,b;c;z)$ is the Gauss–hypergeometric function

(i) For $\alpha=1/2$ the t.p.d. can be calculated:

$$q(x,t\,|\,x_0)=\frac{\mathrm{ch}(a;\mu x)}{\mathrm{ch}(a;\mu x_0)}\frac{e^{t\beta^{2\alpha}}}{\pi}\frac{t(\beta^2+\mu^2)^{1/2}}{[t^2+(x-x_0)^2]^{1/2}}\times K_1((\beta^2+\mu^2)^{1/2}\,[t^2+(x-x_0)^2]^{1/2})$$

We remark that for $\beta=0$, we recover the *Hongler process*.[8]

(ii) For $\alpha = 0$ the expressions are much simpler. The t.p.d. is given by

$$q(x, t \mid x_0) = \frac{\beta}{\sqrt{\pi}\,\Gamma(t)} \frac{\mathrm{ch}(a; \mu x)}{\mathrm{ch}(a; \mu x_0)} \left(\frac{\beta^2 \, |x - x_0|}{2(\beta^2 + \mu^2)^{1/2}} \right)^{t - 1/2} \times K_{t-1/2}((\beta^2 + \mu^2)^{1/2} \, |x - x_0|)$$

the jump rate is

$$W(x \mid x_0) = \frac{\mathrm{ch}(a; \mu x)}{\mathrm{ch}(a; \mu x_0)} \frac{\exp[-(\beta^2 + \mu^2)^{1/2} \, |x - x_0|]}{|x - x_0|}$$

and the conditional moments are, for n odd,

$$\langle [\zeta(t) - x_0]^n \rangle_{x_0} = \frac{2\beta^{2t} n!}{\Gamma((n+1)/2)} \frac{\Gamma((n+1)/2 + t)}{\Gamma(t)} \times \mu \, \mathrm{th}(a; \mu x_0)(\beta^2 + \mu^2)^{-t-(n+1)/2} \times {}_2F_1\left(\frac{n+1}{2} + t, \frac{n}{2} + 1; \frac{3}{2}; \frac{\mu^2}{\beta^2 + \mu^2} \right)$$

and for n even,

$$\langle [\zeta(t) - x_0]^n \rangle_{x_0} = \frac{2\beta^{2t}(n-1)!}{\Gamma(n/2)} \frac{\Gamma(n/2 + t)}{\Gamma(t)} (\beta^2 + \mu^2)^{-t-n/2} \times {}_2F_1\left(\frac{n}{2} + t, \frac{n+1}{2}; \frac{1}{2}; \frac{\mu^2}{\beta^2 + \mu^2} \right)$$

The coefficients of the Kramers–Moyal development simplify to

$$\eta_n(x_0) = \begin{cases} \dfrac{(n-1)! \, \mathrm{th}(a; \mu x_0)}{\beta^{2n}} \times \{ [(\beta^2 + \mu^2)^{1/2} + \mu]^n - [(\beta^2 + \mu^2)^{1/2} - \mu]^n \} & \text{for } n \text{ odd} \\ \dfrac{(n-1)!}{\beta^{2n}} \{ [(\beta^2 + \mu^2)^{1/2} + \mu]^n + [(\beta^2 + \mu^2)^{1/2} - \mu]^n \} & \text{for } n \text{ even} \end{cases}$$

It is interesting to compare this process, which is Markovian, with the non-Markovian process appearing in ref. 9.

APPENDIX A

In this Appendix we deduce, under some regularity conditions, the Kolmogorov–Feller equation (3.2) for a process subordinated to a time-homogeneous diffusion process by an arbitrary proper subordinator.

Theorem 1. Let $p(x, t \mid x_0)$ be the t.p.d. of a Markov process satisfying the following conditions.

(i) For x_0 in a compact set $K \subset \mathbb{R}$ and $\varepsilon > 0$

$$\frac{1}{t} \int_{|x-x_0| \leqslant \varepsilon} dx\, (x-x_0)^m\, p(x, t \mid x_0) = D_m(x_0) + R_m(t, \varepsilon, x_0), \qquad m = 1, 2 \tag{A.1}$$

and

$$\lim_{t \to 0} R_m(t, \varepsilon, x_0) = 0 + o_\varepsilon(1) \qquad \text{uniformly in } x_0 \in K \tag{A.2}$$

(ii) For $|x - x_0| > \varepsilon > 0$, with fixed ε,

$$\frac{1}{t}\, p(x, t \mid x_0) = W(x \mid x_0) + R_W(t, \varepsilon, x, x_0) \tag{A.3}$$

and

$$\lim_{t \to 0} R_W(t, \varepsilon, x, x_0) = 0 \qquad \text{uniformly in } x, x_0 \tag{A.4}$$

(iii) For $t > 0$, $p(x, t \mid x_0)$ is continuously differentiable in t and, if $D_1(x)$, $D_2(x) \neq 0$ twice differentiable in x. The $D_1(x)$ and $D_2(x)$ are once, respectively twice, differentiable.

Then $p(x, t \mid x_0)$ satisfies the equation

$$\frac{\partial}{\partial t} p(x, t \mid x_0) = -\frac{\partial}{\partial x} [D_1(x)\, p(x, t \mid x_0)] + \frac{\partial^2}{\partial x^2} [D_2(x)\, p(x, t \mid x_0)]$$
$$+ \int dy\, [W(x \mid y)\, p(y, t \mid x_0) - W(y \mid x)\, p(x, t \mid x_0)] \tag{A.5}$$

Remarks. The uniformity conditions (A.2) and (A.4) can be stated more explicitly as follows. For x_0 in a compact set $K \subset \mathbb{R}$, there exist functions $\hat{R}_m(t, \varepsilon)$ and $\tilde{R}_m(\varepsilon)$ such that

$$|R_m(t, \varepsilon, x_0)| < \hat{R}_m(t, \varepsilon) \xrightarrow[t \to 0]{} \tilde{R}_m(\varepsilon) \xrightarrow[\varepsilon \to 0]{} 0 \tag{A.6}$$

There is a function $\hat{R}_W(t, \varepsilon)$ such that

$$|R_W(t, \varepsilon, x, x_0)| < \hat{R}_W(t, \varepsilon) \xrightarrow[t \to 0]{} 0 \tag{A.7}$$

The proof follows essentially ref. 3, but with weaker hypotheses needed in the present context.

Proof. We consider test functions $f(x) \in \mathscr{C}_0^3$, with compact support in K, which can be developed in the form

$$f(y) = f(x) + (y-x)\frac{df(x)}{dx} + \frac{1}{2}(y-x)^2\frac{d^2f(x)}{dx^2} + (y-x)^2 R_f(y, x) \qquad \text{(A.8)}$$

where the remainder is equal to

$$R_f(y, x) = (y-x)\frac{1}{3!}\frac{d^3f(\xi)}{dx^3} \qquad \text{with} \quad \xi \in [y, x]$$

We consider

$$\begin{aligned}
&\int dy\, f(y)\frac{\partial}{\partial t}p(y, t \mid x_0) \\
&\quad = \frac{\partial}{\partial t}\int dy\, f(y)\, p(y, t \mid x_0) \\
&\quad = \lim_{\Delta t \to 0}\frac{1}{\Delta t}\int dy\, f(y)[p(y, t+\Delta t \mid x_0) - p(y, t \mid x_0)] \qquad \text{(A.9)}
\end{aligned}$$

Using the Chapman–Kolmogorov equation, separating the range of integration into two parts, and inserting the representation (A.8) in one of them, we can write the first term as

$$\begin{aligned}
&\int dy\, f(y)\, p(y, t+\Delta t \mid x_0) \\
&\quad = \iint_{|x-y|>\varepsilon} dy\, dx\, f(y)\, p(y, \Delta t \mid x)\, p(x, t \mid x_0) \\
&\qquad + \iint_{|x-y|\leqslant\varepsilon} dy\, dx\, f(y)\, p(y, \Delta t \mid x)\, p(x, t \mid x_0) \\
&\quad = \iint_{|x-y|>\varepsilon} dy\, dx\, f(y)\, p(y, \Delta t \mid x)\, p(x, t \mid x_0) \\
&\qquad + \iint_{|x-y|\leqslant\varepsilon} dy\, dx\, f(x)\, p(y, \Delta t \mid x)\, p(x, t \mid x_0) \\
&\qquad + \iint_{|x-y|\leqslant\varepsilon} dy\, dx\left[(y-x)\frac{df(x)}{dx}p(y, \Delta t \mid x)\right. \\
&\qquad \left. + \frac{1}{2}(y-x)^2\frac{d^2f(x)}{dx^2}p(y, \Delta t \mid x)\right] p(x, t \mid x_0) \\
&\qquad + \iint_{\substack{|x-y|\leqslant\varepsilon \\ y \in K}} dy\, dx\, (y-x)^2 R_f(y, x)\, p(y, \Delta t \mid x)\, p(x, t \mid x_0)
\end{aligned}$$

If we multiply the second term of (A.9) by the trivial factor $1=\int dx\, p(x, \Delta t\,|\,y)$ and exchange the dummy arguments x and y, it becomes

$$\int dy\, f(y)\, p(y, t\,|\,x_0)=\iint dy\, dx\, f(x)\, p(y, \Delta t\,|\,x)\, p(x, t\,|\,x_0)$$

Putting these expressions together and using the uniform convergence of the limits (A.2) and (A.4) to exchange the limit $\Delta t \to 0$ and the integrals, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\int dy\, f(y)\, p(y, t\,|\,x_0)\\
&\quad=\int dx\, f(x)\int_{|x-y|>\varepsilon} dy\left\{\left[\lim_{\Delta t\to 0}\frac{1}{\Delta t}\, p(x, \Delta t\,|\,y)\right] p(y, t\,|\,x_0)\right.\\
&\quad\quad\left.-\left[\lim_{\Delta t\to 0}\frac{1}{\Delta t}\, p(y, \Delta t\,|\,x)\right] p(x, t\,|\,x_0)\right\}\\
&\quad\quad+\sum_{m=1}^{2}\int dx\,\frac{1}{m}\frac{d^m f(x)}{dx^m}\\
&\quad\quad\times\left[\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{|x-y|\leqslant\varepsilon} dy\,(y-x)^m\, p(y, \Delta t\,|\,x)\right] p(x, t\,|\,x_0)\\
&\quad\quad+\lim_{\Delta t\to 0}\frac{1}{\Delta t}\iint_{\substack{|x-y|\leqslant\varepsilon\\ y\in K}} dy\, dx\,(y-x)^2\, R_f(y, x)\, p(y, \Delta t\,|\,x)\, p(x, t\,|\,x_0)
\end{aligned}$$

and inserting (A.1)–(A.3), we obtain

$$\begin{aligned}
&=\int dx\, f(x)\int_{|x-y|>\varepsilon} dy\,[W(x\,|\,y)\, p(y, t\,|\,x_0)-W(y\,|\,x)\, p(x, t\,|\,x_0)]\\
&\quad+\int dx\,\frac{df(x)}{dx}\, D_1(x)\, p(x, t\,|\,x_0)\\
&\quad+\int dx\,\frac{1}{2}\frac{d^2 f(x)}{dx^2}\, D_2(x)\, p(x, t\,|\,x_0)\\
&\quad+\sum_{m=1}^{2}\int dx\,\frac{1}{m}\frac{d^m f(x)}{dx^m}\,[\lim_{\Delta t\to 0} R_m(\varepsilon, \Delta t, x)]\, p(x, t\,|\,x_0)\\
&\quad+\lim_{\Delta t\to 0}\frac{1}{\Delta t}\iint_{\substack{|x-y|\leqslant\varepsilon\\ y\in K}} dy\, dx\,(y-x)^2\, R_f(y, x)\, p(y, \Delta t\,|\,x)\, p(x, t\,|\,x_0)
\end{aligned} \tag{A.10}$$

This equality is satisfied for all ε, and we can thus take the limit $\varepsilon \to 0^+$. The three terms involving remainders vanish in this limit: For the first two, using the uniform convergence (A.6) and the fact that the integral is over a compact set, we can write

$$\begin{aligned}
&\left| \int dx \, \frac{d^m f(x)}{dx^m} \left[\lim_{\Delta t \to 0} R_m(\varepsilon, \Delta t, x) \right] p(x, t \mid x_0) \right| \\
&\quad \leqslant \tilde{R}_m(\varepsilon) \int dx \left| \frac{d^m f(x)}{dx^m} \right| p(x, t \mid x_0) \\
&\quad \leqslant c_1 \tilde{R}_m(\varepsilon) \int dx \; p(x, t \mid x_0) \\
&\quad = c_1 \tilde{R}_m(\varepsilon) \xrightarrow[\varepsilon \to 0]{} 0
\end{aligned}$$

For the last term we notice by the same arguments that

$$|R_f(y, x)| = |y - x| \, \frac{1}{3!} \left| \frac{d^3 f(\xi)}{dx^3} \right| \leqslant \varepsilon \, \frac{1}{3!} \left| \frac{d^3 f(\xi)}{dx^3} \right| \leqslant c_2 \varepsilon$$

and therefore

$$\begin{aligned}
&\lim_{\Delta t \to 0} \frac{1}{\Delta t} \left| \iint_{\substack{|x-y| \leqslant \varepsilon \\ y \in K}} dy \, dx \, (y - x)^2 \, R_f(y, x) \, p(y, \Delta t \mid x) \, p(x, t \mid x_0) \right| \\
&\quad \leqslant c_2 \varepsilon \lim_{\Delta t \to 0} \frac{1}{\Delta t} \iint_{\substack{|x-y| \leqslant \varepsilon \\ y \in K}} dy \, dx \, (y - x)^2 \, p(y, \Delta t \mid x) \, p(x, t \mid x_0) \\
&\quad \leqslant c_2 \varepsilon \int_{K'} dx \left[\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{|x-y| \leqslant \varepsilon} dy \, (y - x)^2 \, p(y, \Delta t \mid x) \right] p(x, t \mid x_0) \\
&\quad \leqslant c_2 \varepsilon \int_{K'} dx \, [D_2(x) + \tilde{R}_2(\varepsilon)] \, p(x, t \mid x_0) \\
&\quad \leqslant c_3 \varepsilon \xrightarrow[\varepsilon \to 0]{} 0
\end{aligned}$$

where $K' \supset K$ is compact. The final step is integration by parts in (A.10).

Theorem 2. Let $q(x, t \mid x_0)$ be the t.p.d. of a Markov process subordinated to a diffusion with t.p.d. $p(x, s \mid x_0)$ by a proper subordinator $v(z) = cz + \int_{0^+}^{\infty} d\varrho(s) \, (1 - e^{-sz})$ such that:

(a) $p(x, s \mid x_0)$ satisfies the conditions of Theorem 1, with $W(x \mid x_0) = 0$.

(b) For $t>0$ and $|x-x_0|>\varepsilon>0$, $p(x, t\,|\,x_0)$ is bounded.

(c) For $t>0$, $q(x, t\,|\,x_0)$ is continuously differentiable in t and, if $c\neq 0$, twice differentiable in x.

Then $q(x, t\,|\,x_0)$ satisfies for $t>0$ the equation

$$\begin{aligned}\frac{\partial}{\partial t} q(x, t\,|\,x_0) &= c\left\{-\frac{\partial}{\partial x}\,[D_1(x)\, q(x, t\,|\,x_0)] + \frac{\partial^2}{\partial x^2}\,[D_2(x)\, q(x, t\,|\,x_0)]\right\} \\ &\quad + \fint dy\,[W(x\,|\,y)\, q(y, t\,|\,x_0) - W(y\,|\,x)\, q(x, t\,|\,x_0)] \end{aligned} \tag{A.11}$$

with

$$W(x\,|\,x_0) = \int_{0+}^{\infty} d\varrho(s)\; p(x, s\,|\,x_0)$$

This is a consequence of Lemma 1 and Theorem 1.

Lemma 1. Under the conditions of Theorem 2 we have:

I. For x_0 in a compact set $K\subset\mathbb{R}$ and $\varepsilon>0$ there are functions $S_m(t, \varepsilon, x_0)$, $\hat{S}_m(t, \varepsilon)$, and $\tilde{S}_m(\varepsilon)$ such that

$$\frac{1}{t}\int_{|x-x_0|\leqslant\varepsilon} dx\;(x-x_0)^m\, q(x, t\,|\,x_0) = cD_m(x_0) + S_m(t, \varepsilon, x_0), \quad m=1, 2 \tag{A.12}$$

and

$$|S_m(t, \varepsilon, x_0)| < \hat{S}_m(t, \varepsilon) \xrightarrow[t\to 0]{} \tilde{S}_m(\varepsilon) \xrightarrow[\varepsilon\to 0]{} 0 \tag{A.13}$$

II. For $|x-x_0|>\varepsilon>0$ with fixed ε, there are functions $S_W(t, \varepsilon, x, x_0)$ and $\hat{S}_W(t, \varepsilon)$ such that

$$\frac{1}{t}\, q(x, t\,|\,x_0) = \int_{0+}^{\infty} d\varrho(s)\; p(x, s\,|\,x_0) + S_W(t, \varepsilon, x, x_0) \tag{A.14}$$

and

$$|S_W(t, \varepsilon, x, x_0)| < \hat{S}_W(t, \varepsilon) \xrightarrow[t\to 0]{} 0 \tag{A.15}$$

i.e., one also has the uniform convergence in the subordinated process.

Proof. I. We start by expressing (A.12) as

$$\frac{1}{t}\int_{|x-x_0|\leqslant\varepsilon} dx\,(x-x_0)^m\,q(x,t\,|\,x_0)$$
$$=\frac{1}{t}\int_{|x-x_0|\leqslant\varepsilon} dx\,|x-x_0|^m\int_0^\infty d\gamma_t(s)\,p(x,s\,|\,x_0)$$
$$=\frac{1}{t}\int_0^\infty d\gamma_t(s)\int_{|x-x_0|\leqslant\varepsilon} dx\,(x-x_0)^m\,p(x,s\,|\,x_0) \tag{A.16}$$

The exchange of the integrals is justified since

$$\int_{|x-x_0|\leqslant\varepsilon} dx\,|x-x_0|^m\,q(x,t\,|\,x_0)\leqslant\varepsilon^m \tag{A.17}$$

We split the integral into three pieces:

$$\int_0^\infty=\int_0^{\sqrt{\varepsilon}}+\int_{\sqrt{\varepsilon}}^T+\int_T^\infty \tag{A.18}$$

where $T>0$ is an arbitrary constant. Using (A.1), we can write the first integral as

$$\int_0^{\sqrt{\varepsilon}} d\gamma_t(s)\,\frac{s}{t}\frac{1}{s}\int_{|x-x_0|\leqslant\varepsilon} dx\,(x-x_0)^m\,p(x,s\,|\,x_0)$$
$$=D_m(x_0)\int_0^{\sqrt{\varepsilon}} d\gamma_t(s)\,\frac{s}{t}+\int_0^{\sqrt{\varepsilon}} d\gamma_t(s)\,\frac{s}{t}\,R_m(\varepsilon,s,x_0) \tag{A.19}$$

The first term can be written as

$$D_m(x_0)\int_0^{\sqrt{\varepsilon}} d\gamma_t(s)\,\frac{s}{t}=cD_m(x_0)+D_m(x_0)\left[\int_0^{\sqrt{\varepsilon}} d\gamma_t(s)\,\frac{s}{t}-c\right]$$

where the difference can be bounded in K by

$$\left|D_m(x_0)\left[\int_0^{\sqrt{\varepsilon}} d\gamma_t(s)\,\frac{s}{t}-c\right]\right|\leqslant\sup_{y\in K}|D_m(y)|\left|\int_0^{\sqrt{\varepsilon}} d\gamma_t(s)\,\frac{s}{t}-c\right|$$

The term involving the remainder in (A.19) can be estimated (for ε small enough) using (A.6) as

$$\left|\int_0^{\sqrt{\varepsilon}} d\gamma_t(s)\,\frac{s}{t}\,R_m(\varepsilon,s,x_0)\right|\leqslant\Big[\sup_{s'\in(0,\sqrt{\varepsilon})}\hat{R}_m(s',\varepsilon)\Big]\int_0^{\sqrt{\varepsilon}} d\gamma_t(s)\,\frac{s}{t}$$

For the last two integrals of (A.18), we use (A.17) to obtain the bound

$$\int_{\sqrt{\varepsilon}}^{\infty} d\gamma_t(s) \frac{1}{t} \int_{|x-x_0| \leqslant \varepsilon} dx \, |x-x_0|^m \, p(x, t \mid x_0) \leqslant \varepsilon^m \int_{\sqrt{\varepsilon}}^{\infty} d\gamma_t(s) \frac{1}{t}$$

independently of x_0. Further,

$$\begin{aligned} \varepsilon^m \int_{\sqrt{\varepsilon}}^{T} d\gamma_t(s) \frac{1}{t} &= \varepsilon^{m-1/2} \int_{\sqrt{\varepsilon}}^{T} d\gamma_t(s) \frac{1}{t} \varepsilon^{1/2} \\ &\leqslant \varepsilon^{m-1/2} \int_{\sqrt{\varepsilon}}^{T} d\gamma_t(s) \frac{1}{t} s \leqslant \varepsilon^{m-1/2} \int_{0^+}^{T} d\gamma_t(s) \frac{s}{t} \end{aligned}$$

Putting the preceding estimates together, we can write the following uniform bound:

$$\left| \frac{1}{t} \int_0^{\infty} d\gamma_t(s) \int_{|x-x_0| \leqslant \varepsilon} dx \, (x-x_0)^m \, p(x, s \mid x_0) - cD_m(x_0) \right| \leqslant S_m(t, \varepsilon)$$

with

$$\begin{aligned} S_m(t, \varepsilon) \doteq{}& \sup_{y \in K} |D_m(y)| \left| \int_0^{\sqrt{\varepsilon}} d\gamma_t(s) \frac{s}{t} - c \right| \\ &+ [\sup_{s' \in (0^+, \sqrt{\varepsilon})} \hat{R}_m(s', \varepsilon)] \int_0^{\sqrt{\varepsilon}} d\gamma_t(s) \frac{s}{t} \\ &+ \varepsilon^m \int_T^{\infty} d\gamma_t(s) \frac{1}{t} + \varepsilon^{m-1/2} \int_{0^+}^{T} d\gamma_t(s) \frac{s}{t} \end{aligned}$$

We can now easily verify that $\lim_{\varepsilon \to 0} \lim_{t \to 0} S_m(t, \varepsilon) = 0$: Using Eqs. (B.10) and (B.4) of Appendix B, we obtain

$$\begin{aligned} \lim_{t \to 0} S_m(t, \varepsilon) ={}& \sup_{y \in K} |D_m(y)| \left| \int_0^{\sqrt{\varepsilon}} d\sigma(s) - c \right| \\ &+ [\sup_{s' \in (0, \sqrt{\varepsilon})} \hat{R}_m(s', \varepsilon)] \int_0^{\sqrt{\varepsilon}} d\sigma(s) \\ &+ \varepsilon^m \int_T^{\infty} d\varrho(s) + \varepsilon^{m-1/2} \int_{0^+}^{T} d\varrho(s) \, s \end{aligned} \tag{A.20}$$

The integrability condition (2.12) on $d\varrho(s)$ guarantees that the integrals are finite. Finally, invoking (B.5), one sees that (A.20) vanishes in the limit $\varepsilon \to 0$.

II. The expression (A.14) for the jump rate can be obtained as follows. Since

$$\frac{1}{t} q(x, t \mid x_0) = \frac{1}{t} \int_0^\infty d\gamma_t(s)\, p(x, s \mid x_0)$$

the remainder term of (A.14) can be written as

$$\begin{aligned} S_W(t, \varepsilon, x, x_0) &= \int_0^\infty d\gamma_t(s) \frac{1}{t} p(x, s \mid x_0) - \int_{0^+}^\infty d\varrho(s)\, p(x, s \mid x_0) \\ &= \int_\tau^\infty \left[d\gamma_t(s) \frac{1}{t} - d\varrho(s) \right] p(x, s \mid x_0) \\ &\quad + \int_0^\tau d\gamma_t(s) \frac{s}{t} \frac{p(x, s \mid x_0)}{s} - \int_{0^+}^\tau d\varrho(s)\, s \frac{p(x, s \mid x_0)}{s} \end{aligned}$$

where $\tau > 0$ is an arbitrary constant. The first integral can be bounded by

$$\begin{aligned} &\left| \int_\tau^\infty \left[d\gamma_t(s) \frac{1}{t} - d\varrho(s) \right] p(x, s \mid x_0) \right| \\ &\qquad \leqslant \Big[\sup_{\substack{s' \in (0^+, \infty) \\ |x - x_0| > \varepsilon}} p(x, s' \mid x_0) \Big] \int_\tau^\infty \left| d\gamma_t(s) \frac{1}{t} - d\varrho(s) \right| \end{aligned}$$

The supremum is finite according to the condition (b) of the theorem. For the second and third integrals we use (A.3) for the special case of a diffusion, and obtain the bound

$$\begin{aligned} &\left| \int_0^\tau d\gamma_t(s) \frac{s}{t} \frac{p(x, s \mid x_0)}{s} \right| + \left| \int_{0^+}^\tau d\varrho(s)\, s \frac{p(x, s \mid x_0)}{s} \right| \\ &\quad = \left| \int_0^\tau d\gamma_t(s) \frac{s}{t} R_W(s, \varepsilon, x, x_0) \right| + \left| \int_{0^+}^\tau d\varrho(s)\, s R_W(s, \varepsilon, x, x_0) \right| \\ &\quad \leqslant \Big[\sup_{s' \in [0, \tau]} \hat{R}_W(s', \varepsilon) \Big] \left[\int_0^\tau d\gamma_t(s) \frac{s}{t} + \int_{0^+}^\tau d\varrho(s)\, s \right] \end{aligned}$$

We can now put the preceding estimates together to obtain

$$\begin{aligned} &|S_W(t, \varepsilon, x, x_0)| \\ &\quad = \left| \int_0^\infty d\gamma_t(s) \frac{1}{t} p(x, s \mid x_0) - \int_{0^+}^\infty d\varrho(s)\, p(x, s \mid x_0) \right| \end{aligned}$$

$$\leqslant [\sup_{\substack{s' \in (0^+, \infty) \\ |x - x_0| > \varepsilon}} p(x, s \mid x_0)] \int_\tau^\infty \left[d\gamma_t(s) \frac{1}{t} - d\varrho(s) \right]$$

$$+ [\sup_{s' \in [0, \tau]} \hat{R}_W(s', \varepsilon)] \left[\int_0^\tau d\gamma_t(s) \frac{s}{t} + \int_{0^+}^\tau d\varrho(s)\, s \right]$$

$$\doteq \hat{S}_W(t, \varepsilon)$$

If we take the limit $t \to 0$, the first term vanishes due to (B.1), and the last two terms give

$$\lim_{t \to 0} \hat{S}_W(t, \varepsilon) = [\sup_{s' \in [0, \tau]} \hat{R}_W(s', \varepsilon)] \left[\int_0^\tau d\sigma(s) + \int_{0^+}^\tau d\sigma(s) \right]$$

Since τ is arbitrary and it does not appear on the left-hand side, the inequality is still valid in the limit $\tau \to 0^+$. In this limit the terms in the curly brackets tend to c [by Eq. (B.5)] and the term in the square brackets tends to zero.

APPENDIX B

In this Appendix we discuss some properties and relations connecting $d\gamma_t(s)$, $v(z)$, and $d\varrho(s)$, and deduce Eq. (3.4). We will use the step function

$$\theta(s) = \begin{cases} 0 & \text{for } s < 0 \\ 1 & \text{for } s \geqslant 0 \end{cases}$$

Lemma. (i) The measure $d\gamma_t(s)$ is related to $d\varrho(s)$ by the following weak limit:

$$d\varrho(s) = \lim_{t \to 0^+} \frac{1}{t} d\gamma_t(s) \qquad \text{in} \quad (0^+, \infty) \tag{B.1}$$

i.e., $d\varrho(s)$ is the jump rate of the subordinator.

(ii) The subordinator has a drift equal to c.

(iii) It has no diffusion.

Proof. We will use the following facts[7]: Since, according to (2.10), $v'(z) \equiv dv/dz$ is completely monotone, it has a positive inverse Laplace transform $d\sigma(s)$:

$$v'(z) = \int_0^\infty d\sigma(s)\, e^{-sz} \tag{B.2}$$

By integration one obtains the representation (2.11) for $v(z)$:

$$v(z) = cz + \int_{0^+}^{\infty} d\sigma(s) \frac{1}{s} (1 - e^{-sz}) \tag{B.3}$$

i.e.,

$$d\varrho(s) = \frac{d\sigma(s)}{s} \quad \text{in} \quad (0^+, \infty) \tag{B.4}$$

and

$$c = \sigma(0^+) - \sigma(0) \tag{B.5}$$

(i) We will prove (B.1) using the uniqueness of the inverse Laplace transform: From (2.7) we have

$$e^{-tv(z)} = \int_0^{\infty} d\gamma_t(s)\, e^{-sz} \qquad \text{for} \quad z \geqslant 0 \tag{B.6}$$

By differentiation with respect to z we obtain

$$v'(z)\, e^{-tv(z)} = \int_0^{\infty} d\gamma_t(s) \frac{s}{t} e^{-sz} \tag{B.7}$$

and therefore

$$v'(z) = \lim_{t \to 0} \int_0^{\infty} d\gamma_t(s) \frac{s}{t} e^{-sz} \qquad \text{for} \quad z \geqslant 0 \tag{B.8}$$

Comparing with (B.2), we obtain

$$\lim_{t \to 0} \int_0^{\infty} d\gamma_t(s) \frac{s}{t} e^{-sz} = \int_0^{\infty} d\sigma(s)\, e^{-sz} \qquad \text{for} \quad z > 0 \tag{B.9}$$

By the continuity theorem of P. Lévy (see ref. 24), this implies

$$\lim_{t \to 0} d\gamma_t(s) \frac{s}{t} = d\sigma(s) \qquad \text{in} \quad [0, \infty) \tag{B.10}$$

and thus (B.1).

(ii) The drift is given by

$$\lim_{\varepsilon \to 0} \lim_{t \to 0} \frac{1}{t} \int_0^{\varepsilon} d\gamma_t(s)\, s = \lim_{\varepsilon \to 0} \int_0^{\varepsilon} d\sigma(s) = c$$

(iii) The diffusion coefficient is given by

$$\lim_{\varepsilon \to 0} \lim_{t \to 0} \frac{1}{t} \int_0^\varepsilon d\gamma_t(s)\, s^2 = \lim_{\varepsilon \to 0} \int_0^\varepsilon d\sigma(s)\, s \leqslant \lim_{\varepsilon \to 0} \varepsilon \int_0^\varepsilon d\sigma(s) = 0$$

Lemma. The subordinator measures $d\gamma_t^{(0)}(s)$ and $d\gamma_t^{(c)}(s)$ corresponding, respectively, to a subordinator $\nu^{(0)}(z)$ without a linear part (i.e., $c = 0$) and to $\nu^{(c)}(z) = cz + \nu^{(0)}(z)$ are related by

$$d\gamma_t^{(c)}(s) = \theta(s - ct)\, d\gamma_t^{(0)}(s - ct) \tag{B.11}$$

Proof. It follows immediately from (2.7) and

$$\begin{aligned} &\int_0^\infty d\gamma_t^{(0)}(s - ct)\, \theta(s - ct)\, e^{-zs} \\ &\quad = \int_0^\infty d\gamma_t^{(0)}(r)\, e^{-zr} e^{-zct} = e^{-[c + \nu^{(0)}(z)]t} \end{aligned} \tag{B.12}$$

Corollary. If $q^{(0)}(x, t \mid x_0)$ is the subordinated t.p.d. to $p(x, t \mid x_0)$ through $q^{(0)}(x, t \mid x_0) = \int_0^\infty d\gamma_t^{(0)}(s)\, p(x, s \mid x_0)$, then the process subordinated through $d\gamma_t^{(c)}(s)$ has t.p.d.

$$\begin{aligned} q^{(c)}(x, t \mid x_0) &= \int_0^\infty d\gamma_t^{(0)}(s - ct)\, \theta(s - ct)\, p(x, s \mid x_0) \\ &= \int_0^\infty d\gamma_t^{(0)}(r)\, p(x, r + ct \mid x_0) \end{aligned} \tag{B.13}$$

Corollary. The t.p.d. $q^{(c)}$ and $q^{(0)}$ are related by both of the following equations:

$$\begin{aligned} q^{(c)}(x, t \mid x_0) &= \int_{-\infty}^\infty dy\, p(x, ct \mid y)\, q^{(0)}(y, t \mid x_0) \\ &= \int_{-\infty}^\infty dy\, q^{(0)}(x, t \mid y)\, p(y, ct \mid x_0) \end{aligned} \tag{B.14}$$

Proof. Using the two forms of the Chapman–Kolmogorov equation

$$p(x, s + ct \mid x_0) = \begin{cases} \displaystyle\int_{-\infty}^\infty dy\, p(x, ct \mid y)\, p(y, s \mid x_0) \\ \displaystyle\int_{-\infty}^\infty dy\, p(x, s \mid y)\, p(y, ct \mid x_0) \end{cases} \tag{B.15}$$

we can write

$$q^{(c)}(x, t \mid x_0) = \int_0^{\infty} d\gamma_t^{(0)}(s)\, p(x, s + ct \mid x_0)$$

$$= \begin{cases} \displaystyle\int_{-\infty}^{\infty} dy\, p(x, ct \mid y) \int_0^{\infty} d\gamma_t^{(0)}(s)\, p(y, s \mid x_0) \\ \displaystyle\int_{-\infty}^{\infty} dy \left[\int_0^{\infty} d\gamma_t^{(0)}(s)\, p(x, s \mid y) \right] p(y, ct \mid x_0) \end{cases}$$

$$= \begin{cases} \displaystyle\int_{-\infty}^{\infty} dy\, p(x, ct \mid y)\, q^{(0)}(y, t \mid x_0) \\ \displaystyle\int_{-\infty}^{\infty} dy\, q^{(0)}(x, t \mid y)\, p(y, ct \mid x_0) \end{cases} \tag{B.16}$$

which is (B.14).

APPENDIX C

In this Appendix we deduce the Kramers–Moyal expansion for a process subordinated to the Wiener process and relate its coefficients to the *Taylor expansion* at zero of the analytic subordinator $v(z)$.

The Kramers–Moyal expansion is a formal way to write the Kolmogorov–Feller equation (1.1) as a linear partial differential equation of infinite order:

$$\frac{\partial}{\partial t} q(x, t \mid x_0) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial x^n} [\eta_n(x)\, q(x, t \mid x_0)]$$

The coefficients $\eta_n(y)$, if they exist, are given by[1]

$$\eta_n(y) = \lim_{t \to 0^+} \frac{1}{t} \int_{-\infty}^{\infty} dx\, (x - x_0)^n\, q(x, t \mid y) = \int_{-\infty}^{\infty} dx\, (x - x_0)^n\, W(x \mid y) \tag{C.1}$$

(provided that one can exchange the limit and the integral).

From the pseudodifferential equation (4.5), we deduce formally an expansion

$$\frac{\partial}{\partial t} q(x, t \mid x_0) = -v\left(-\frac{1}{2} \frac{\partial^2}{\partial x^2} \right) q(x, t \mid x_0)$$

$$= -\sum_{n=1}^{\infty} \frac{v^{[n']}(0)}{n!} \left(-\frac{1}{2} \frac{\partial^2}{\partial x^2} \right)^n q(x, t \mid x_0)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, v^{[n']}(0)}{2^n n!} \frac{\partial^{2n}}{\partial x^{2n}} q(x, t \mid x_0) \tag{C.2}$$

where we have denoted $v^{[k']}(z) = (d^k/dz^k)\, v(z)$. We notice that if $v(z)$ is analytic, we obtain

$$v^{[n']}(0) = c\delta_{1,n} + (-1)^{n-1} \int_{0^+}^{\infty} \varrho(dr)\, r^n$$

Thus, (C.2) can be written as

$$\frac{\partial}{\partial t} q(x, t \mid x_0) = \frac{c}{2} \frac{\partial^2}{\partial x^2} q(x, t \mid x_0) + \sum_{n=1}^{\infty} \frac{1}{2^n n!} \left[\int_{0^+}^{\infty} \varrho(dr)\, r^n \right] \times \frac{\partial^{2n}}{\partial x^{2n}} q(x, t \mid x_0)$$

On the other hand, if we compute the Kramers–Moyal coefficients, using (C.1), we find that

$$\begin{aligned} \eta_n(x) &= \int_{-\infty}^{\infty} dz\, (z-x)^n\, W(z \mid x) \\ &= \int_{0^+}^{\infty} \varrho(dr) \int_{-\infty}^{\infty} dz\, (z-x)^n \frac{e^{-(z-x)^2/2r}}{(2\pi r)^{1/2}} \\ &= \begin{cases} 0 & \text{for } n \text{ odd} \\ \dfrac{(2n)!}{2^n n!} \displaystyle\int_{0^+}^{\infty} \varrho(dr)\, r^{n/2} + c\delta_{2,n} & \text{for } n \text{ even} \end{cases} \end{aligned}$$

[A sufficient condition for the validity of these calculations is $\int_{0^+}^{\infty} \varrho(dr)\, r^{1/2} < \infty$.] The two expansions are thus identical.

ACKNOWLEDGMENTS

We thank M.-O. Hongler, who introduced us to the method of subordination, and W. Amrein for very helpful discussions. This work was partially supported by the Swiss National Science Foundation.

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